Lecture 3

Compound Propositions, Logical Equivalence

Biconditional Statements

Definition: Let p and q be propositions. The **biconditional statement**, denoted by $p \leftrightarrow q$, is the statement "p if and only if q".

$p \leftrightarrow q$ is meant to be "If p, then q and if q, then p".

$p \leftrightarrow q$ is true when both p and q have the same truth value and is false otherwise.

Truth Table of biconditional statement ($\leftrightarrow$)

p	q	$p \rightarrow q$	$q \rightarrow p$	$p \leftrightarrow q$
F	F	T	T	T
F	T	T	F	F
T	F	F	T	F
T	T	T	T	T

Biconditional Statements

Example:

$p =$ Sam can take the flight.

$q =$ Sam buys a ticket.

$p \leftrightarrow q =$ Sam can take the flight **if and only if** Sam buys a ticket

Some more ways to express $p \leftrightarrow q$.

- p is necessary and sufficient for q. $(\because p \leftrightarrow q \equiv (p \to q) \land (q \to p))$
- p iff q.

Constructing "Bigger" Compound Propositions

Repeatedly apply logical operator $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$ on propositions to create bigger propositions.

Example: p $\quad q$ $\quad \neg q$ $\quad p \vee \neg q$ $\quad p \wedge q$ $\quad (p \vee \neg q) \rightarrow (p \wedge q)$

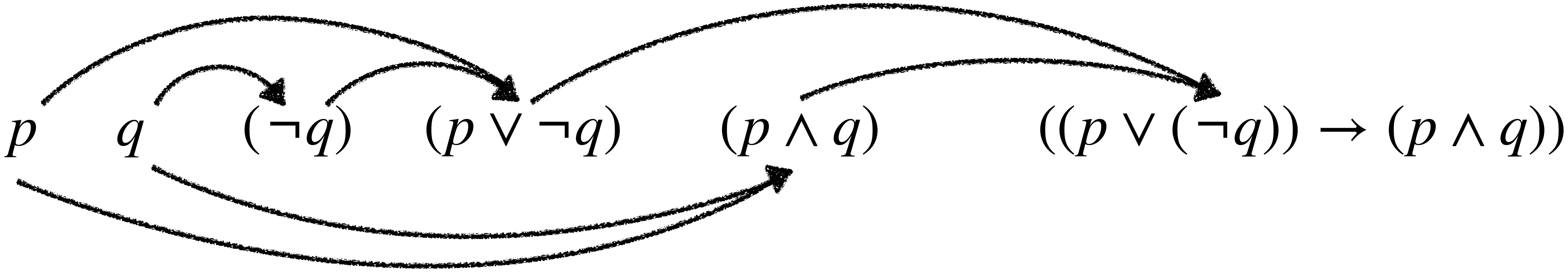
Truth table of
$(p \vee \neg q) \rightarrow (p \wedge q)$

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
F	F	T	T	F	F
F	T	F	F	F	T
T	F	T	T	F	F
T	T	F	T	T	T

Precedence of Logical Operators

Tip: Using parentheses with every application of logical operator specifies the order in which logical operators in a compound proposition are to applied.

Example:



When parentheses are not present, then the order in which logical operator are applied can be determined from the following **order of precedence**:

$\neg$	$\wedge$	$\vee$	$\rightarrow$	$\leftrightarrow$
1	2	3	4	5

Precedence of Logical Operators

Order of precedence:

$\neg$	$\wedge$	$\vee$	$\rightarrow$	$\leftrightarrow$
1	2	3	4	5

Example:

$p \vee q \wedge r$ is $p \vee (q \wedge r)$ not $(p \vee q) \wedge r$.

$p \vee q \rightarrow r$ is $(p \vee q) \rightarrow r$ not $p \vee (q \rightarrow r)$.

$p \vee \neg q \rightarrow p \wedge q$ is $(p \vee (\neg q)) \rightarrow (p \wedge q)$ not $p \vee (((\neg q) \rightarrow p) \wedge q)$.

Tautology and Contradiction

Will use the term “compound proposition” for expressions formed from propositional variables and logical operators such as $(p \vee q) \rightarrow \neg r$, $\neg p \rightarrow (q \wedge r)$, etc.

Some special compound propositions.

- A compound proposition that is **always true** irrespective of the truth values of the propositional variables is called a **tautology**, such as $p \vee \neg p$.
- A compound proposition that is **always false** irrespective of the truth values of the propositional variables is called a **contradiction**, such as $p \wedge \neg p$.
- A compound proposition that is neither a tautology nor a contradiction is called a **contingency**, such as $p \wedge q$.

Logical Equivalence

Compound propositions that have the same truth values in all possible cases are called **logically equivalent**.

Definition: Compound propositions p and q containing same set of propositional variables are **logically equivalent**, denoted by $p \equiv q$, if $p \leftrightarrow q$ is a tautology.

Example: Show that $\neg p \vee q \equiv p \rightarrow q$

p	q	$\neg p$	$\neg p \vee q$	$p \leftrightarrow q$
F	F	T	T	T
F	T	T	T	T
T	F	F	F	F
T	T	F	T	T

Logical Equivalence

Example: Show that $\neg(p \vee q) \equiv \neg p \wedge \neg q$

p	q	$\neg p$	$\neg q$	$p \vee q$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
F	F	T	T	F	T	T
F	T	T	F	T	F	F
T	F	F	T	T	F	F
T	T	F	F	T	F	F

DeMorgan's Law:

1. $\neg(p \vee q) \equiv \neg p \wedge \neg q$
2. $\neg(p \wedge q) \equiv \neg p \vee \neg q$

Important Logical Equivalences

Identity Laws: $p \wedge T \equiv p$, $p \vee F \equiv p$

Domination Laws: $p \vee T \equiv T$, $p \wedge F \equiv F$

Idempotent Laws: $p \wedge p \equiv p$, $p \vee p \equiv p$

De Morgan's Laws: $\neg(p \vee q) \equiv \neg p \wedge \neg q$, $\neg(p \wedge q) \equiv \neg p \vee \neg q$

Double Negation Law: $\neg(\neg p) \equiv p$

Commutative Laws: $p \wedge q \equiv q \wedge p$, $p \vee q \equiv q \vee p$

Note: $p, q, r,$ etc. can be compound propositions.

For instance, $\neg((p \wedge q) \vee (s \rightarrow t)) \equiv \neg(p \wedge q) \wedge \neg(s \rightarrow t)$

Important Logical Equivalences

Absorption Law:

$$p \lor (p \land q) \equiv p$$
$$p \land (p \lor q) \equiv p$$

Negation Laws:

$$p \land \neg p \equiv F$$
$$p \lor \neg p \equiv T$$

Associative Laws:

$$(p \lor q) \lor r \equiv p \lor (q \lor r)$$
$$(p \land q) \land r \equiv p \land (q \land r)$$

Distributive Laws:

$$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$$
$$p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$$